An improved criterion for Kapitza’s pendulum stability

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Abstract

An enhanced and more exact criterion for dynamic stabilization of the parametrically driven inverted pendulum is obtained: the boundaries of stability are determined with greater precision and are valid in a wider region of the system parameters than previous results. The lower boundary of stability is associated with the phenomenon of subharmonic resonances in this system. The relationship of the upper limit of dynamic stabilization of the inverted pendulum with ordinary parametric resonance (i.e., with destabilization of the lower equilibrium position) is established. Computer simulation of the physical system aids the analytical investigation and proves the theoretical results.

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1 Introduction

A well-known curiosity of classical mechanics is the dynamic stabilization of an inverted pendulum. An ordinary rigid planar pendulum whose pivot is forced to oscillate along the vertical line becomes stable in the inverted position if the driving amplitude and frequency lie in certain intervals. At small and moderate deviations from the inverted position the pendulum shows no tendency to turn down. Being deviated, the pendulum executes relatively slow oscillations about the vertical line on the background of rapid oscillations of the suspension point. Due to friction these slow oscillations gradually damp away, and the pendulum eventually comes to the vertical inverted position.

This type of dynamic stability was probably first pointed out by Stephenson [1] more than a century ago. In 1951 such extraordinary behaviour of the pendulum was physically explained and investigated experimentally in detail by Pjotr Kapitza [2], and the corresponding physical device is now widely known as ‘Kapitza’s pendulum.’ Since then this intriguing system has attracted the attention of many researchers, and the theory of the phenomenon may seem to be well elaborated — see, for example, [3]. A vast list of references on the subject can be found in [4]. Nevertheless, more and more new features in the behaviour of this apparently inexhaustible system are reported regularly.
Among new discoveries regarding the inverted pendulum, the most important for determination of the stability boundaries are the destabilization of the (dynamically stabilized) inverted position at large driving amplitudes through excitation of period-2 (‘flutter’) oscillations [5]–[6], and the existence of \( n \)-periodic ‘multiple-nodding’ regular oscillations [7]. The relationship of ‘flutter’ oscillations in the inverted pendulum with ordinary parametric resonance of the hanging pendulum is discussed in [8]. A physical interpretation of periodic multiple-nodding oscillations as subharmonic parametric resonances is given in [9].

In this paper we present an improved criterion for dynamic stabilization of the parametrically driven inverted pendulum. The lower boundary of the parameter space region in which the inverted pendulum can be stable is associated with conditions of subharmonic resonance of an infinitely large order. The upper boundary is related to some branch of excitation of ordinary parametric resonance. These relationships allow us to determine the boundaries in a wider region of the system parameters (including relatively low frequencies and large amplitudes of excitation), and with a greater precision compared to previous results.

2 The physical system

We consider for simplicity a light rigid rod of length \( l \) with a heavy small bob of mass \( m \) on its end and assume that all the mass of the pendulum is concentrated in the bob. The force of gravity \( mg \) provides a restoring torque \(-mgl \sin \varphi\) whose value is proportional to the sine of angular deflection \( \varphi \) of the pendulum from the lower equilibrium position. With the suspension point at rest, this torque makes the deviated pendulum to swing about the lower stable equilibrium position. When the axis of the pendulum is constrained to move with acceleration along the vertical line, it is convenient to analyze the motion in the non-inertial reference frame associated with this axis. Due to the acceleration of this frame of reference, there is one more force exerted on the pendulum, namely the pseudo force of inertia \(-m\ddot{z}\), where \( z(t) \) is the vertical coordinate of the axis. The torque of this force \(-m\ddot{z} \sin \varphi\) must be added to the torque of the gravitational force.

We assume that the axis is forced to execute a given harmonic oscillation along the vertical line with a frequency \( \omega \) and an amplitude \( a \), i.e., the motion of the axis is described by the following equation:

\[
  z(t) = a \cos \omega t \quad \text{or} \quad z(t) = a \sin \omega t. \tag{1}
\]

Depending on the problem in question, in (1) the first or the second choice of the initial phase of the pivot oscillations may occur more convenient. The force of inertia \( F_{\text{in}}(t) \) exerted on the bob in the non-inertial frame of reference also has the same sinusoidal dependence on time:

\[
  F_{\text{in}}(t) = -m \frac{d^2z(t)}{dt^2} = -m\ddot{z}(t) = m\omega^2 z(t). \tag{2}
\]

This force of inertia is directed upward during the time intervals for which \( z(t) > 0 \), i.e., when the axis is over the middle point of its oscillations. We see this directly from equation (2) for \( F_{\text{in}}(t) \), whose right-hand side depends on time exactly as \( z \)-coordinate of the axis. Therefore during the corresponding half-period of the
oscillation of the pivot this additional force is equivalent to some weakening of the force of gravity. During the second half-period the axis is below its middle position \((z(t) < 0)\), and the action of this additional force is equivalent to some strengthening of the gravitational force.

The graphs of the time history and the phase trajectories for different modes of the pendulum that are presented further in this paper are obtained by a numerical integration of the exact differential equation for angular deflection \(\varphi(t)\) of the pendulum with the oscillating pivot. This equation includes, besides the torque \(-mg\sin\varphi\) of the gravitational force \(mg\) (here \(g\) is the free fall acceleration), the torque of the force of inertia \(F_{in}(t)\) which depends explicitly on time \(t\):

\[
\ddot{\varphi} + 2\gamma \dot{\varphi} + \left(\frac{g}{l} - \frac{a}{l} \omega^2 \cos \omega t\right) \sin \varphi = 0. \tag{3}
\]

The second term in (3) originates from the frictional torque which is assumed in this model to be proportional to the momentary value of the angular velocity \(\dot{\varphi}\). Damping constant \(\gamma\) in this term is inversely proportional to quality factor \(Q\) which is conventionally used to characterize damping of small natural oscillations under viscous friction: \(Q = \omega_0/2\gamma\), where \(\omega_0 = \sqrt{g/l}\) is the frequency of infinitely small natural oscillations in the absence of pivot oscillations.

Oscillations about the inverted position can be formally described by the same differential equation (3) with negative values of \(g\), as if the force of gravity were directed upward. In other words, we can treat the free fall acceleration \(g\) in equation (3) as a control parameter whose variation is physically equivalent to variation of the gravitational force. When this control parameter \(g\) is diminished through zero to negative values, the gravitational torque in (3) also reduces down to zero and then changes its sign to the opposite. Such a ‘gravity’ tends to bring the pendulum into the inverted position \(\varphi = \pi\), destabilizing the lower equilibrium position \(\varphi = 0\) of the unforced pendulum and making upper position \(\varphi = \pi\) stable: at \(g < 0\) the inverted position in (3) is equivalent to the lower position at positive \(g\).

3 An approximate criterion of stability

Since the first works of Kapitza [2], stabilization of the inverted pendulum is usually explained by the method of averaging based on an heuristic approach, in which the pendulum motion can be represented as a relatively slow oscillation described by a smooth function \(\psi(t)\). The slow motion is distorted by superposition of rather small fast vibrations \(\delta(t) \ll 1\) of a high frequency:

\[
\varphi(t) = \psi(t) + \delta(t) \approx \psi(t) - \frac{z(t)}{l} \sin \psi = \psi(t) - \frac{a}{l} \sin \psi \cos \omega t. \tag{4}
\]

The rapidly varying component \(\delta(t)\) in the above expression for \(\varphi(t)\) is obtained on the assumption that during one period \(T\) of fast vibration of the pivot the low component \(\psi\) remains almost constant. In this approximation the time dependence of \(\delta(t)\) in (4) can be easily seen from figure 1: when the oscillating pivot \(A\) is displaced from its mid-point through some distance \(z\) for a given angle \(\psi\), the momentary angle \(\varphi\) acquires an additional value of approximately \(-(z/l)\sin \psi\).

Figure 1 shows the pendulum at the utmost displacements of the pivot in the inertial (upper part \(a\)) and non-inertial (lower part \(b\)) frames. In the non-inertial
Figure 1: Behaviour of the pendulum with an oscillating axis in the inertial reference frame (a) and in the frame associated with the pivot (b).

frame associated with the pivot, the bob of the pendulum moves up and down along an arc of a circle and occurs in positions 1 and 2 at the instants at which the oscillating axis reaches its extreme positions 1 and 2 respectively, shown in the upper part of figure 1. We note that the rod has the same simultaneous orientations in both reference frames at instant 1 as well as at instant 2. When the pivot is displaced downward to position 2 from its midpoint, the force of inertia $F_2$ exerted on the bob is also directed downward. In the other extreme position 1 the force of inertia $F_1$ has an equal magnitude and is directed upward. However, the torque of the force of inertia in position 1 is greater than in position 2 because the arm of the force in this position is greater. Therefore on average the force of inertia creates a torque about the axis that tends to turn the pendulum upward, into the vertical inverted position, in which the rod is parallel to the direction of oscillations.

This qualitative explanation of the phenomenon of dynamic stabilization can be supported by approximate quantitative considerations. The momentary torque of the force of inertia $F_{in}(t)$ is given by the product of this force (2) and its arm $l \sin \varphi(t) \approx l \sin \psi + l \cos \psi \delta(t)$. Averaging this torque over the period $T$ of rapid oscillations of the pivot, we can regard the slow angular coordinate $\psi(t)$ to be constant: $\psi(t) \approx \tilde{\psi}$. In this approximation the first term of the mean torque $\langle F_{in}(t) l \sin \psi \rangle$ vanishes due to the zero average value $\langle F_{in}(t) \rangle$ of the force of inertia, while the second term gives

$$\langle F_{in}(t) l \cos \psi \delta(t) \rangle = -ma^2 \omega^2 \sin \psi \cos \tilde{\psi} \langle \cos^2 \omega t \rangle = -\frac{1}{2}ma^2 \omega^2 \sin \psi \cos \tilde{\psi}. \quad (5)$$

This mean torque of the force of inertia at $\psi > \pi/2$ tends to bring the pendulum into the inverted position. It can exceed in magnitude the torque $-mgl \sin \psi$ of the gravitational force that tends to tip the pendulum down when the following condition is fulfilled:

$$a^2 \omega^2 > 2gl, \quad \text{or} \quad \frac{a}{l} \cdot \frac{\omega}{\omega_0} > \sqrt{2}, \quad \text{or} \quad \frac{a}{l} > \sqrt{-2k}, \quad (6)$$

where $\omega_0 = \sqrt{g/l}$ is the frequency of infinitely small natural oscillations of the pendulum. This is the commonly known (approximate) criterion of the inverted
pendulum stability. In the perspective of the forthcoming generalization and improvement, we have also expressed in (6) this criterion in terms of parameter \( k \), which is defined by the following expression:

\[
k = \frac{\omega_0^2}{\omega^2} = \frac{g}{l\omega^2}.
\]  

(7)

This dimensionless parameter \( k \) (inverse normalized drive frequency squared), being physically less meaningful than \( \omega/\omega_0 \), is nevertheless more convenient for further investigation, because the improved criterion acquires more simple form in terms of \( k \). Negative values of \( k \) correspond to negative \( g \) values, which can be formally assigned to the inverted pendulum, as we have already noted earlier.

Another widely used approach to the problem of the inverted pendulum stability is based on the concept of an effective potential. Indeed, the slow component of the pendulum motion \( \psi(t) \) can be represented conveniently as a movement of some particle in an effective potential described by function \( U = U(\psi) \) which takes into account the torque of the gravitational force and the mean torque of the force of inertia (2):

\[
U(\psi) = U_{\text{gr}}(\psi) + U_{\text{in}}(\psi) = mgl(1 - \cos \psi) + \frac{1}{4}ma^2\omega^2(1 - \cos 2\psi).
\]  

(8)

Figure 2: Graphs of the gravitational potential energy \( U_{\text{gr}}(\psi) \), mean potential energy \( U_{\text{in}}(\psi) \) in the field of the force of inertia, and of the total potential energy \( U(\psi) = U_{\text{tot}}(\psi) \) for the pendulum with an oscillating pivot. The graphs correspond to \((a/l)(\omega/\omega_0) = 2.2\).

The graphs of \( U_{\text{gr}}(\psi) \) and \( U_{\text{in}}(\psi) \) are shown in figure 2. They both have a sinusoidal shape, but the period of \( U_{\text{in}}(\psi) \) is just one half of the period of \( U_{\text{gr}}(\psi) \). Their minima at \( \psi = 0 \) coincide, thus generating the principal minimum of the total potential function \( U(\psi) = U_{\text{tot}}(\psi) \). This minimum corresponds to the stable lower equilibrium position of the pendulum. But the next minimum of \( U_{\text{in}}(\psi) \) is located at \( \psi = \pi \), where \( U_{\text{gr}}(\psi) \) has its maximum corresponding to the inverted position of the pendulum.

If the criterion of stability (6) for the inverted pendulum is fulfilled, total potential \( U = U(\psi) \), besides the global minimum at \( \psi = 0 \) that corresponds to the lower equilibrium position, has additional local minima at \( \psi = \pm \pi \). These minima
correspond to the inverted position. Oscillations of the particle locked in such a minimum describe the smooth motion $\psi(t)$ of the pendulum (with $\pi - \psi < \theta_{\text{max}}$, see figure 2) in the vicinity of dynamically stabilized inverted position.

The differential equation for the particle in potential $U = U(\psi)$, that is, for the slow component of the pendulum motion, can be written as follows:

$$\ddot{\psi} = -\omega_0^2 \sin \psi - \frac{1}{2 \ell^2} \omega^2 \cos \psi \sin \psi.$$  

(9)

Slow small oscillations occurring near the bottom of each well of the potential $U = U(\psi)$ are almost harmonic. The frequencies $\omega_{\text{down}}$ and $\omega_{\text{up}}$ of oscillations about the lower ($\psi = 0$) and upper ($\psi = \pm \pi$) positions respectively are given by the following expressions (see [8] for detail):

$$\omega_{\text{down}}^2 = \frac{a^2 \omega_0^2}{2 \ell^2} + \omega_0^2, \quad \omega_{\text{up}}^2 = \frac{a^2 \omega_0^2}{2 \ell^2} - \omega_0^2.$$  

(10)

The approach based on separation of slow and rapid components of the pendulum motion is applicable for relatively small amplitudes and high frequencies of the pivot oscillations. Hence the commonly known expressions (6) for stability criteria are valid only at $\omega/\omega_0 \gg 1$ and $a/l \ll 1$.

4 Subharmonic resonances of high orders and stability of the inverted pendulum

An alternative way to obtain the stability criterion for the inverted pendulum is based on the relationship between damping oscillations about the inverted position and stationary regimes of subharmonic oscillations. This approach allows us to find a more exact and enhanced criterion which is valid in a wider than (6) region of the system parameters. In particular, this improved criterion can be used for relatively large amplitudes and low frequencies of the pivot oscillations, for which the method of averaging is inapplicable.

When the driving amplitude and frequency lie within certain ranges, the pendulum, instead of gradually approaching the equilibrium position (either dynamically stabilized inverted position or ordinary downward position) by the process of damped slow oscillations, can be trapped in a $n$-periodic limit cycle. By virtue of a ‘phase locking’ (definite phase relationship between the phases of oscillations of the pendulum and the pivot), the pendulum is regularly fed by additional energy to compensate for frictional losses. The phase trajectory exactly repeats itself after $n$ periods $T$ of excitation are finished. One period of such non-damping oscillations of the pendulum equals exactly an integer number $n$ cycles of the pivot vibrations. The frequency $\omega/n$ of the principal harmonic equals $1/n$ of the excitation frequency $\omega$. This allows us to call this phenomenon a subharmonic resonance of order $n$.

For the inverted pendulum with a vibrating pivot, periodic oscillations of this type were first described by Acheson [7], who called them ‘multiple-nodding’ oscillations. Computer simulations show that the pendulum motion in this regime reminds some kind of an original dance. Similar ‘dancing’ oscillations can be executed also (at appropriate values of the driving parameters) about the ordinary equilibrium position. ‘Multiple-nodding’ oscillations can occur also in the absence of gravity.
about any of the two equivalent dynamically stabilized equilibrium positions [8]. An example of such period-8 oscillations is shown in figure 3.

![Figure 3: The spatial path, phase orbit with Poincaré sections, and graphs of large-amplitude stationary period-8 oscillations. The fundamental harmonic with the frequency \( \omega/8 \) dominates the spectrum. This harmonic describes the smooth (slow) motion of the pendulum. The most important high harmonics have frequencies \( 7\omega/8 \) and \( 9\omega/8 \). At large swing the third harmonic (frequency \( 3\omega/8 \)) is noticeable. This spectral component reflects the non-harmonic character of slow oscillations in the non-parabolic well of the effective potential.](image)

A simple approximate derivation of conditions in which multiple-nodding oscillations exist can be based on the effective potential concept. The natural slow oscillatory motion in the effective potential well (figure 2) is almost periodic (exactly periodic in the absence of friction). A subharmonic oscillation of order \( n \) can occur if one cycle of this slow motion covers approximately \( n \) driving periods, that is, when the driving frequency \( \omega \) is close to an integer multiple \( n \) of the natural frequency of slow oscillations near either the inverted or the ordinary equilibrium position. For small amplitudes of the slow oscillations, each of the minima of the effective potential can be approximated by a parabolic well in which the smooth component of motion is almost harmonic. Equating \( \omega_{\text{up}} \) or \( \omega_{\text{down}} \) (10) to \( \omega/n \), we find the threshold (low-amplitude) conditions for the subharmonic resonance of order \( n \):

\[
m_{\text{min}} = \sqrt{2(1/n^2 - k)},
\]

where \( k \) is given by (7). As we already indicated above, negative values of the parameter \( k \) can be treated as referring to the inverted pendulum. The limit of this expression at \( n \rightarrow \infty \) gives the same approximate criterion (6) of stability for the inverted pendulum: \( m_{\text{min}} = \sqrt{-2k} \) (where \( k < 0 \)).

Being based on a decomposition of motion on slow oscillations and rapid vibrations with the driving frequency, equation (11) is valid if the amplitude of constrained vibration of the axis is small compared to the pendulum length \( (a \ll l) \), and the
driving frequency is much greater than the frequency of small natural oscillations of
unforced pendulum ($\omega \gg \omega_0$). These restrictions mean that we should not expect
from the discussed here approach to give an exhaustive description of the paramet-
rically driven pendulum in all cases. In particular, this approach cannot explain
why the lower position of the pendulum becomes unstable within certain ranges of
the system parameters (in the intervals of parametric instability). It cannot explain
also the destabilization of the dynamically stabilized inverted position through ex-
citation of ‘flutter’ oscillations. Next we develop a different approach that is free of
the above-mentioned restrictions.

5 Conditions of subharmonic resonances

An example of stationary subharmonic oscillations whose period equals 8 periods
of the pivot is shown in figure 4. The graphs show angular velocity $\dot{\phi}(t)$ time
dependence with the graphs of its harmonics, and the spectrum of velocity.

The fundamental harmonic whose period equals eight driving periods dominates
the spectrum. We may treat it as a subharmonic (as an ‘undertone’) of the driving
oscillation. This principal harmonic of the frequency $\omega/n$ describes the smooth
component $\psi(t)$ of the compound period-8 oscillation:

$$\psi(t) = A \sin(\omega t/n).$$

The threshold conditions for stabilization of the inverted pendulum correspond
to a slow oscillation of an indefinitely long period about $\psi = \pi$. Such an oscillation
can be regarded as a subharmonic oscillation of the inverted pendulum whose order
$n$ tends to infinity. Therefore the desired criterion of stability can be found from
the conditions of $n$-order subharmonic resonance in the limit $n \to \infty$. Next we try
to find these conditions for the subharmonic resonance of arbitrary order $n$, valid,
in particular, for relatively low frequencies and large amplitudes of the excitation.

When the drive amplitude is slightly greater then the threshold for the subhar-
monic resonance of order $n$, the pendulum swing is small, and we can replace $\sin \psi
by $\psi$ in expression (4) for the momentary angle of deflection $\varphi(t)$:

$$\varphi(t) \approx \psi(t) - \frac{a}{l} \sin \psi(t) \cos \omega t \approx \psi(t) - m \psi(t) \cos \omega t, \quad m = \frac{a}{l}. \quad (12)$$
This means that the spectrum of $n$-periodic oscillations of a small amplitude consists of the principal harmonic $A \sin(\omega t/n)$ with the frequency $\omega/n$ and two higher harmonics of order $n-1$ and $n+1$ with equal amplitudes:

$$\varphi(t) = A \sin(\omega t/n) - mA \sin(\omega t/n) \cos \omega t$$
$$= A \sin(\omega t/n) + \frac{mA}{2} \sin\left(\frac{n-1}{n} \omega t\right) - \frac{mA}{2} \sin\left(\frac{n+1}{n} \omega t\right).$$ (13)

This spectral composition is in general supported by the plots in figure 4 obtained by numerical integration of the exact differential equation (3). It may seem strange that the harmonic component of order $n$ with the frequency of excitation is absent in the spectrum. However, this peculiarity is easily explained. Indeed, the rapid component $\cos \omega t$ enters $\varphi(t)$ being multiplied by $\sin \psi(t)$, that is, it has a slow varying amplitude that changes sign each time the pendulum crosses the equilibrium position. This means that the oscillation with frequency $\omega$ is not a harmonic of $\varphi(t)$, because harmonics of a periodic function must have constant amplitudes.

When the pendulum crosses the equilibrium position, the high harmonics of orders $n-1$ and $n+1$ add in the opposite phases and almost do not distort the graph of the smooth motion (described by the principal harmonic). Near the utmost deflections of the pendulum the phases of both high harmonics coincide, and distort noticeably the slow component.

According to equation (13), both high harmonics have equal amplitudes $(m/2)A$. However, we see from the plots in figure 4 that these amplitudes are slightly different. Therefore we can try to improve the approximate solution for $\varphi(t)$, equation (13), as well as the theoretical threshold values of $m$ for excitation of subharmonic resonances, by assuming for an improved solution a similar spectrum but with unequal amplitudes, $A_{n-1}$ and $A_{n+1}$, of the two high harmonics (for $n > 2$, the case of $n = 2$ will be considered separately). Moreover, figure 4 shows small contributions in $\varphi(t)$ of harmonics with frequencies $(2n-1)\omega/n$ and $(2n+1)\omega/n$, which we also include in the trial function:

$$\varphi(t) = A_1 \sin(\omega t/n) + A_{n-1} \sin[(n-1)\omega t/n] + A_{n+1} \sin[(n+1)\omega t/n] +$$
$$+ A_{2n-1} \sin[(2n-1)\omega t/n] + A_{2n+1} \sin[(2n+1)\omega t/n].$$ (14)

Since oscillations at the threshold of excitation have infinitely small amplitudes, we can use instead of equation (3) the following linearized (Mathieu) equation:

$$\ddot{\varphi} + 2\gamma \dot{\varphi} + \omega^2(k - m \cos \omega t) \varphi = 0, \quad (m = a/l).$$ (15)

Here we have used parameter $k = g/(l\omega^2)$ introduced by (7). Substituting $\varphi(t)$, equation (14), into this equation (with $\gamma = 0$) and expanding the products of trigonometric functions, we obtain the following system of approximate equations for the coefficients $A_1$, $A_{n-1}$ and $A_{n+1}$, $A_{2n-1}$ and $A_{2n+1}$:

$$2(kn^2 - 1)A_1 + mn^2A_{n-1} - mn^2A_{n+1} = 0,$$
$$mn^2A_1 + 2[n^2(k - 1) + 2n - 1]A_{n-1} - mn^2A_{2n-1} = 0,$$
$$-mn^2A_1 + 2[n^2(k - 1) - 2n - 1]A_{n+1} + mn^2A_{2n+1} = 0,$$
$$mn^2A_{n-1} + 2[n^2(k - 4) + 4n - 1]A_{2n-1} = 0,$$
$$mn^2A_{n+1} + 2[n^2(k - 4) - 4n - 1]A_{2n+1} = 0.$$ (16)
The homogeneous system (16) has a nontrivial solution if its determinant equals zero. This condition yields an equation (not cited here) for the corresponding threshold (minimal) normalized driving amplitude \( m_{\text{min}} = a_{\text{min}}/l \) at which \( n \)-periodic mode \( \varphi(t) \) given by expression (14) can exist. This equation can be solved with the help of, say, Mathematica package by Wolfram Research, Inc.). Then, after substituting this critical driving amplitude \( m_{\text{min}} \) into (16), fractional amplitudes \( A_{n-1}/A_1 \), \( A_{n+1}/A_1 \), \( A_{2n-1}/A_1 \) and \( A_{2n+1}/A_1 \) of high harmonics for a given order \( n \) can be found as the solutions to the homogeneous system of equations (16).

![Figure 5:](image)

**Figure 5:** The normalized driving amplitude \( m = a/l \) versus \( k = (\omega_0/\omega)^2 \) (inverse normalized driving frequency squared) at the lower boundary of the dynamic stabilization of the inverted pendulum (the left curve marked as \( n \to \infty \)), and at subharmonic resonances of several orders \( n \) (see text for detail).

The final expressions for \( m_{\text{min}} \) and the amplitudes of harmonics are too bulky to be cited here. We have used them in figure 5 for plotting the curves of \( m_{\text{min}} \) as functions of \( k = (\omega_0/\omega)^2 \) (inverse normalized driving frequency squared) corresponding to subharmonic oscillations of different orders \( n \) (thin curves).

To verify our analytical results for subharmonic oscillations in a computer simulation, we choose \( k = -0.3 \), corresponding to the drive frequency \( \omega = 1.826 \omega_0 \), for which the separation of slow and rapid motions is obviously inapplicable. The above-described calculation for the subharmonic oscillation of order \( n = 8 \) predicts for the threshold normalized drive amplitude \( m_{\text{min}} = a_{\text{min}}/l \) a value 87.73% of the pendulum length. The simulation presented in figure 6 perfectly confirms this theoretical prediction.

As we already noted, the criterion of stability for the inverted pendulum can be
related with the condition of subharmonic resonances of infinitely large order $n$ in the vicinity of $\varphi = \pm \pi$. Hence the limit of $m_{\text{min}}$ at $n \to \infty$ gives an improved formula for the lower boundary of the dynamic stabilization of the inverted pendulum:

$$m_{\text{min}} = 2 \sqrt{\frac{k(k-1)(k-4)}{3k-8}}, \quad (17)$$

instead of the commonly known approximate criterion $m_{\text{min}} = \sqrt{-2k}$, given by equation (7). The minimal amplitude $m_{\text{min}} = a_{\text{min}}/l$ that corresponds to the improved criterion (17) of dynamic stabilization is shown as a function of $k = (\omega_0/\omega)^2$ by the thick left curve marked as $n \to \infty$ in figure 5. This curve is localized wholly in the region of negative $k$ values. To compare the improved criterion (17) with the commonly known criterion (7) of the inverted pendulum stability, the latter is also shown in figure 5 by the thin curve (red in the electronic version). We note how these two curves diverge dramatically at low frequencies and large amplitudes of the pivot oscillations.

The other curves to the right from this boundary show the dependence on $k$ of minimal driving amplitudes for which the subharmonic resonances of several orders can exist (the first curve for $n = 6$, and the others for $n$ values diminishing down to $n = 2$ from left to right). At negative $k$ values these curves give the threshold drive amplitudes for subharmonic oscillations about the inverted position. Case $k = 0$ corresponds to zero gravity (or infinitely high drive frequency). Points of intersection of the curves with the ordinate axis on this diagram give minimal drive amplitudes for which in the absence of gravity subharmonic oscillations of certain order $n$ can exist about any of the two dynamically stabilized positions (figure 3 shows an example of such period-8 oscillations).
Continuations of the curves to positive $k$ values correspond to subharmonic parametric resonances (‘multiple-nodding’ oscillations) about the downward equilibrium position. The curve for $n = 2$ corresponds to ordinary parametric resonance, in which two cycles of excitation take place during one full oscillation of the pendulum. In the absence of friction the threshold drive amplitude for this resonance tends to zero at $\omega \to \omega_0/2$, that is, at $k \to 1/4$. From figure 5 we see clearly that the curve, corresponding to $n = 2$ subharmonic oscillations of the inverted pendulum (‘flutter’ mode), and parametric resonance of ordinary (hanging) pendulum belong to the essentially same branch of period-2 regular behaviour. In $k < 0$ region this branch gives the upper boundary of dynamic stabilization for the inverted pendulum (see Section 6).

![Phase trajectories and time histories](image)

Figure 7: Phase trajectories and time histories of a gradually damping oscillations about the inverted position just over the lower and just below the upper boundaries of dynamic stabilization.

We note that the existence of subharmonic oscillations does not disprove criterion (17) of the inverted pendulum stability. Indeed, the pendulum is trapped in $n$-periodic limit cycle (with $n > 2$) only if the initial state belongs to a certain small basin of attraction that corresponds to this limit cycle. Otherwise the pendulum eventually comes to rest in the inverted position.

For the drive frequency $\omega = 1.826 \omega_0$ ($k = -0.3$) improved criterion (17) gives for the lower boundary of stability the drive amplitude $a_{\text{min}} = 0.868l$. The upper boundary (see Section 6) at $k = -0.3$ equals $a_{\text{max}} = 0.929l$. Computer simulations (figure 7) show how at $a = 0.875l$ (over the lower boundary) and at $a = 0.925l$ (below the upper boundary) the pendulum, initially deflected through 10°, returns gradually to the inverted position. At smaller than 0.868 $l$ values of the drive amplitude the inverted pendulum is unstable. Figure 8 shows how at $k = -0.3$ and $a = 0.80l$ the pendulum, being released at only 1° deflection from the inverted position, occurs eventually in a chaotic regime (‘tumbling’ chaos). The graphs in
figure 8 show the initial stage of the time history. The set of Poincaré sections in the phase plane gives impression of the further random behaviour. We note that the inverted pendulum at these drive parameters should be stable according to conventional criterion (6), which at $k = -0.3$ gives $a_{\text{min}} = 0.775 l$.

Figure 8: Chaotic oscillations below the boundary of stability.

6 The upper boundary of dynamic stabilization

The curve $n = 2$ in figure 5 corresponds to the upper boundary of dynamic stabilization for the inverted pendulum: after a disturbance the pendulum does not come to rest in the up position, no matter how small the release angle, but instead eventually settles into a limit cycle, executing finite amplitude steady-state oscillation (about the inverted vertical position). The period of such an oscillation is twice the driving period, and its swing grows as the excess of the drive amplitude over the threshold $a_{\text{max}}$ is increased.

This loss of stability of the inverted pendulum has been first described in 1992 by Blackburn et al. [5] and demonstrated experimentally in [6]. The authors [5] called these limit-cycle oscillations the 'flutter' mode. Because the flutter mode and parametric resonance belong to the same branch of period-2 stationary regime, the same analytical method can be used to calculate conditions of their excitation. Simulations show a very simple spectral composition for both, namely a superposition of the fundamental harmonic whose frequency $\omega/2$ equals half the driving frequency, the third harmonic with the frequency $3\omega/2$, and maybe a tiny admixture of the fifth harmonic:

$$\varphi(t) = A_1 \cos(\omega t/2) + A_3 \cos(3\omega t/2) + A_5 \cos(5\omega t/2).$$

(18)

The phases of harmonics in (18) correspond to pivot oscillations in the form $z(t) = a \cos \omega t$. Substituting $\varphi(t)$ into the differential equation (15) with $\gamma = 0$ and expanding the products of trigonometric functions, we obtain an expression, in which we should equate to zero the coefficients of $\cos(\omega t/2)$, $\cos(3\omega t/2)$, and
\[
\cos(5\omega t/2). \text{ Thus we get a system of homogeneous equations for the coefficients } A_1, A_3 \text{ and } A_5:
\]
\[
(4k - 2m - 1)A_1 - 2mA_3 = 0,
-2A_1 + (4k - 9)A_3 - 2mA_5 = 0,
-2mA_3 + (4k - 25)A_5 = 0,
\]
which has a nontrivial solution when its determinant equals zero. If we neglect the contribution of the 5th harmonic in \( \varphi(t) \), equation (18), that is, let \( A_5 = 0 \), we get the following approximate expression for the upper boundary of stability:
\[
m_{\text{max}} = \frac{1}{4} \left[ \sqrt{(4k - 9)(20k - 13)} + 4k - 9 \right].
\]

If the 5th harmonic is included, the requirement for non-trivial solution to the system (19) yields a cubic equation for the desired normalized critical driving amplitude \( a_{\text{max}}/l = m_{\text{max}} \). The relevant root of this equation (too cumbersome to be shown here) is used for plotting the curve \( n = 2 \) in figure 5. However, for the interval of \( k \) values under consideration (\(-0.8 \rightarrow 0.6\)) the approximate expression (20) gives a curve which is practically indistinguishable from the curve \( n = 2 \) in figure 5.

The curve \( n = 2 \) intersects the ordinate axis at \( m \approx 3(\sqrt{13} - 3)/4 = 0.454 \). This case \((k = 0)\) corresponds to the above-mentioned limit of a very high driving frequency \((\omega/\omega_0 \rightarrow \infty)\) or zero gravity \((\omega_0 = 0)\), so that \( m = 0.454 \) gives the upper limit of stability for each of the two dynamically stabilized equivalent equilibrium positions: if \( m > 0.454 \) at \( g = 0 \), the flutter mode is excited.

The lower and upper boundaries of the dynamical stability gradually converge while the drive frequency is reduced: figure 5 shows that the interval between \( m_{\text{min}} \) and \( m_{\text{max}} \) shrinks to the left, when \( |k| \) is increased. Both boundaries merge at \( k \approx -1.41 \) \((\omega \approx 0.8423\omega_0)\) and \( m \approx 2.451 \). The diminishing island of the inverted state stability vanishes in the surrounding sea of chaotic motions.

For experimental verification of the improved criterion (17) and its comparison with the conventional one (7), we now choose \( k = -0.5 \) (drive frequency \( \omega = 1.4142\omega_0 \)). For this \( k \) value criterion (17) yields \( m_{\text{min}} = 1.1920 \) \((119.20\% \text{ of the pendulum length})\), while (7) gives \( m_{\text{min}} = 1.00 \) \((a_{\text{min}} = l) - 100\% \). The theoretical value for the upper boundary at \( k = -0.5 \) is \( m_{\text{max}} = 1.2226 \) \((122.26\%),\) while according to (20) \( m_{\text{max}} = 1.2265 \) \((122.65\%)\). Simulations show that below

![Figure 9: Dynamical stability at low frequency and large amplitude of the drive.](image)
$m = 115.68\%$ the motion is chaotic (‘tumbling’ chaos), at $m = 115.69\% - 119.19\%$ the pendulum, after a long chaotic transient, is trapped in period-1 non-uniform unidirectional rotation (in contradiction with conventional criterion (7), which predicts stability of the inverted position), and only in the interval $m = 119.20\% - 122.27\%$ the pendulum, being released at a small deviation from the inverted position, eventually comes to rest, in exact accordance with the improved criterion (17). Then, over the upper boundary of stability, at $m = 122.28\% - 123.02\%$ the pendulum occurs in a ‘flutter’ mode; at $m = 123.03\% - 147.01\%$ executes unidirectional rotation; at $m = 147.02\% - 150.6\%$ the pendulum, after a long transient, comes to period-2 oscillation about the inverted position with an amplitude of approximately $260^\circ$ (similar to the oscillation shown in figure 10, page 15); at $m \geq 150.7\%$ the pendulum eventually settles in the unidirectional rotation.

Figure 10: Period-2 oscillation of large amplitude about the inverted position: the phase orbit, spatial trajectory, and graphs of angular velocity $\dot{\phi}(t)$ and angle $\phi(t)$ time dependencies (with the graphs of separate harmonics).

Further simulations refer to the case $k = -1$ ($\omega = \omega_0$). The theoretical values for the lower and upper boundaries of stability for this frequency of the pivot oscillations are $m_{\text{min}} = 1.9069$ (190.69\% of the pendulum length) and $m_{\text{max}} = 1.9138$ (191.38\%), respectively. Figure 9 shows how in this narrow interval the pendulum, being released at $178^\circ$, gradually approaches the inverted position by the process of irregular damping oscillations after about $6^\circ$ maximum angular excursion. The basin of attraction for equilibrium in the inverted position is rather small: at slightly different initial conditions the pendulum, after a long transient, occurs in a large-amplitude (about $255^\circ$) period-2 oscillation about the inverted position. The angular excursion of the pendulum from one extreme position to the other takes one period of excitation and is greater than a full circle. Otherwise we can treat this regime as alternating clockwise and counterclockwise revolutions. The phase orbit and the spatial trajectory of the pendulum bob in figure 10 give an impression of such an extraordinary motion. Just over the upper boundary of stability (191.38\%) the pendulum eventually settles into the ‘flutter’ mode (figure 11).
Figure 11: Flutter oscillations of the inverted pendulum just over the upper boundary of stability.

7 Concluding remarks

The commonly known criterion for dynamic stabilization of the inverted pendulum is usually derived by separation of rapid and slow motions of the pendulum. This approach and related concept of the effective potential for the slow motion are very useful for physical explanation of the dynamic stabilization, as well as of the origin of subharmonic resonances of high orders. However, this separation of rapid and slow motions is admissible only at sufficiently high frequency and small amplitude of the pivot oscillations.

An improved criterion for the lower boundary of dynamic stabilization, valid in a wider region of frequencies and amplitudes of the pivot oscillations, including values for which the method of separation of rapid and slow motions is inapplicable, is obtained by establishing a close relationship between the phenomenon of dynamic stabilization of the inverted pendulum and subharmonic resonances.

The upper boundary of stabilization is related to excitation of period-2 ‘flutter’ oscillation, which is a complete analog of the principal parametric resonance. Both phenomena belong to the same branch of stationary period-2 oscillations, and the criterion for destabilization of the inverted position is obtained by the same method as for excitation of ordinary parametric resonance. Results of corresponding analytical calculations agree perfectly well with computer simulations.

References


